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NON-REGULAR ULTRAFILTERS

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ABSTRACT

We construct non-regular ultrafilters, extending filters which are dual to dense or layered ideals.

Introduction

If there is a σ-complete uniform ultrafilter on a cardinal κ, then κ is greater than or equal to a measurable cardinal. Interested in ultrafilters on smaller sets, one has to look at characteristics, which are weaker than completeness. In this paper we shall consider non-regular ultrafilters.

Definition 1: A filter F on a τ^+-complete Boolean algebra B is (τ,κ)**-regular** iff there is an $A \subseteq F$, $|A| = \kappa$ such that $\prod A' = 0$ for each $A' \subseteq A$, $|A'| = \tau$. ∎

Thus F is non-(τ,κ)-regular if for all $A \subseteq F$, $|A| = \kappa$ there is an $A' \subseteq A$, $|A'| = \tau$ such that $\prod A' \neq 0$.

For the construction of the non-regular ultrafilters we will use ideals, which have strong saturation properties.

Definition 2: An ideal I on a Boolean algebra B is κ**-dense** iff B/I has a dense subset of size $\leq \kappa$. I is κ**-layered** (see [FMSh2]) iff there is a stationary set $S \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)\}$ and some continuous increasing chain of Boolean algebras $\langle B_\alpha \mid \alpha < \kappa^+\rangle$ such that $B/I = \bigcup_{\alpha<\kappa^+} B_\alpha$ and for all $\alpha \in S$ B_α is a

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κ-complete regular subalgebra of B such that $|B_\alpha| \leq \kappa$. I is **strongly κ-layered** if we can choose $S = \{\alpha < \kappa^+ |\ \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)\}$. ∎

Note that κ-dense or κ-layered ideals are κ^+-saturated.

Two important results about the existence of non-regular ultrafilters are from Laver and from Foreman, Magidor and Shelah. In [L], Laver constructed a non-(ω, ω_1)-regular, uniform ultrafilter on ω_1. He extended the dual filter of an ω_1-dense, ω_1-complete, normal ideal using $\diamond_{\omega_1}$ (or at least CH as a result of [BSV]). In [FMSh2], Foreman, Magidor and Shelah got a non-(τ, κ)-regular, uniform ultrafilter on $\kappa = \tau^+$, τ regular, by forcing with a κ^+-distributive partial ordering over a model with $\diamondsuit_\kappa$ and with a κ-layered, normal ideal on κ.

In this paper we improve these results and give a more general method to construct non-regular ultrafilters on suitable sets without using $\diamondsuit_\kappa$. We shall prove the following theorem:

THEOREM: *Let $\kappa > \omega$ be regular, let X be a nonempty set. Suppose that $I \subseteq \mathcal{P}(X)$ is a κ-complete, normal ideal on X such that $\{x \in X |\ \alpha \in x\} \in I^*$ for all $\alpha < \kappa$. If*

(1) *I is κ-dense or*

(2) *$\Box_\kappa$ and I is strongly κ-layered,*

then there is an ultrafilter $U \supseteq I^*$ on X, which is non-(τ, κ)-regular for all $\tau < \kappa$ such that $\{x \in X |\ x \cap \kappa$ is $<\tau$-closed$\} \in I^*$.

We shall actually prove a slightly more general theorem, only talking about ultrafilters and ideals on Boolean algebras. Therefore we shall later introduce some notion of normality for ideals on Boolean algebras.

Note that $\{x \in X |\ x \cap \kappa$ is $< \tau$-closed$\} \in I^*$ is trivial if $\tau = \omega$. It is also true, if $X = \kappa = \tau^+$, τ regular and if I is normal and κ^+-saturated, since then $\{\alpha < \kappa |\ \mathrm{cf}(\alpha) = \tau\} \in I^*$ (Shelah [Sh]). So Laver's result is a special case of this theorem. The theorem also implies the result of Forman, Magidor and Shelah: We can force a κ-layered ideal on κ with a κ^+-distributive partial ordering to become strongly κ-layered. Then by forcing with another κ^+-distributive partial ordering (which therefore do not destroy strongly κ-layeredness on κ) we have $\Box_\kappa$.

How to get dense or layered ideals? If κ is a huge cardinal and if $\tau < \kappa$ is regular, then there is a generic extension, in which $\kappa = \tau^+$ and there is a strongly κ-layered, κ-complete ideal on κ [FMSh2]. Starting with an almost huge cardinal, Woodin constructed an ω_1-dense, ω_1-complete ideal on ω_1.

We can apply this theorem to limit cardinals as well. Starting with a measurable cardinal, Kunen and Paris [KP] constructed a generic extension with a κ^+-saturated, κ-complete, normal, uniform ideal on a weakly compact, non-measurable cardinal κ. Looking at the proof, one can see, that the ideal is actually κ-dense and that $\{\alpha < \kappa | \operatorname{cf}(\alpha) > \tau\} \in I^\star$ for all $\tau < \kappa$. Thus we have an ultrafilter on a weakly compact, non-measurable cardinal κ, which is non-(τ, κ)-regular for all $\tau < \kappa$ (Corollary 16).

A filter U on κ is called **regular**, if U is (ω, κ)-regular. It is well-known, that ultrapowers with regular ultrafilters have maximal size, i.e. $|A^\kappa/U| = |A|^\kappa$ for any infinite set A. Starting with an ω_1-dense or strongly ω_1-layered ideal on ω_1, Laver [L] and Shelah [FMSh2] got uniform ultrafilters U on ω_1 such that $|\omega^{\omega_1}/U| = \aleph_1$. They used CH or $\diamondsuit_{\omega_1}$. The construction in this paper needs no cardinal arithmetic assumptions and yields $|\omega^{\omega_1}/U| = 2^{\aleph_0}$ (Corollary 11).

For completeness note that in L all uniform ultrafilters are regular: Prikry [P] showed that every uniform ultrafilter on κ^+ is (κ, κ^+)-regular if $V = L$. Ketonen [Ke] weakened the assumption $V = L$ to $\neg 0^\#$, Jensen [DJK] to $\neg L^\mu$, i.e. there is no inner model with a measurable cardinal. Jensen proved further, that in L every uniform ultrafilter on ω_n $(n < \omega)$ is regular. Finally Donder [D] showed, that in L every uniform ultrafilter on any cardinal is regular.

Notation

Let **On** denote the class of all Ordinals, **Lim**, **Succ**, **Card** the classes of all limit ordinals, successor ordinals and cardinals respectively. For all A, B let ${}^A B := \{f | f\colon A \to B\}$. For any cardinal τ $[A]^\tau$ and $[A]^{<\tau}$ denotes the set of all subsets of A of power τ and of power $<\tau$ respectively. A is $<\tau$-closed if for all $x \in [A]^{<\tau}$ $\bigcup x \in A$. $\langle a_\alpha | \alpha < \beta\rangle$ is **continuous increasing**, if $a_\alpha \subseteq a_{\alpha'}$ for all $\alpha \leq \alpha'$ and $a_\gamma = \bigcup_{\alpha<\gamma} a_\alpha$ for all $\gamma \in \mathrm{Lim}$. We write **Ba** for Boolean algebra and **uf** for ultrafilter. Let $\langle B, +, \cdot, -, 0, 1\rangle$ be a Ba. B is κ-**complete**, if $\sum A$ exists in B for all $A \in [B]^{<\kappa}$. B is (κ, τ)-**distributive** iff $\prod_{\alpha<\overline{\kappa}} \sum_{\beta<\overline{\tau}} u_{\alpha\beta} = \sum_{f\colon \overline{\kappa} \to \overline{\tau}} \prod_{\alpha<\overline{\kappa}} u_{\alpha f(\alpha)}$ for all $u_{\alpha\beta} \in B$, $\alpha < \overline{\kappa} < \kappa$, $\beta < \overline{\tau} < \tau$. $A \subseteq B$ has the **finite intersection property (fip)** iff $\prod A' \neq 0$ for all finite $A' \subseteq A$. If $A \subseteq B$ then $A^\star := \{-a | a \in A\}$ is the **dual** set to A. For every $U \subseteq B$ with the fip we write $U^+ := \{b \in B | U \cup \{b\}$ has the fip$\}$. For ideals $I \subseteq B$ let $I^+ := (I^\star)^+$, i.e. $I^+ = B \setminus I$. $B^+ := \{0\}^+ = B \setminus \{0\}$. An ideal $I \subseteq B$ is called κ-**saturated** iff every antichain in B/I has cardinality less then κ. If B is κ-complete, then we call I **κ-complete** iff $\sum A \in I$ for all $A \in [I]^{<\kappa}$. A subalgebra C of B is a **regular subalgebra** if every maximal antichain A in C is also a maximal antichain in B. This is fulfilled iff for each $b \in B^+$ there is a $c \in C^+$ such that for all $c' \in C^+$, if $c' \leq c$ then $c' \cdot b \neq 0$. (c is called a **projection** of b).

Let $I \subseteq \mathcal{P}(X)$ be an ideal on a set X. I is **fine** iff for every $i \in \bigcup X$ $\{x \in X \mid i \in x\} \in I^\star$. I is called **normal** iff whenever $f\colon X \to V$ is regressive on some $b \in I^+$ (i.e. $f(x) \in x$ for all $x \in b$), then there is a $y \in \operatorname{range} f$ such that $\{x \in b \mid f(x) = y\} \in I^+$. This is equivalent to I being closed under diagonal unions, i.e. if for all $i \in \bigcup X$ $X_i \in I$, then $\nabla_{i \in \cup X} X_i = \{x \in X \mid \exists i \in x\ x \in X_i\} \in I$. I is **uniform** iff for every $A \in I^\star$ $|A| = |X|$. For filters we use the same notations as for ideals in their analogous meanings.

Dense ideals and non-regular ultrafilters

We introduce some notion of normality and fineness for ideals on Boolean algebras.

Definition 3: Let B be a κ^+-complete Boolean algebra and let $A = \langle a_\alpha \mid \alpha < \kappa \rangle \in {}^\kappa B$. We call an ideal $I \subseteq B$ **A-fine** iff $a_\alpha \in I^\star$ for all $\alpha < \kappa$. I is **A-normal** iff for all $\langle b_\alpha \mid \alpha < \kappa \rangle \in {}^\kappa I$ $\sum_{\alpha<\kappa}(a_\alpha \cdot b_\alpha) \in I$.

Remark 4: An A-fine ideal $I \subseteq B$ is A-normal iff for all $\langle b_\alpha \mid \alpha < \kappa \rangle \in {}^\kappa B$ $\sum_{\alpha<\kappa}[b_\alpha]_I = [\sum_{\alpha<\kappa} a_\alpha b_\alpha]_I$ in the algebra B/I. Let $I \subseteq \mathcal{P}(X)$ be an ideal on some set X, let $\langle i_\alpha \mid \alpha < \kappa \rangle$ be any 1-1 sequence, let $a_\alpha := \{x \in X \mid i_\alpha \in x\}$ and $A := \langle a_\alpha \mid \alpha < \kappa \rangle$. If I is normal, then I is A-normal. If I is A-normal and $\bigcup X \subseteq \{i_\alpha \mid \alpha < \kappa\}$, then I is normal. ∎

Definition 5: Let B be a Boolean algebra, let $A, C, D \subseteq B$. We call C a **D-cover of A** iff for all $a \in A \cap D$ there is a $c \in C$ such that $a \cdot c \in D$. ∎

We shall later use this notion to formulate some covering property of ultrafilters, which is sufficient to get non-regularity.

For the proof of the next lemma we apply a method similar to one, which is used in [BSV] for the construction of ultrafilters without socalled nowhere dense towers.

LEMMA 6: *Let $\kappa > \omega$ be regular, let I be an ideal on a κ-complete Boolean algebra B. Then for all $A \in [B]^{\leq\kappa}$ there exists an ultrafilter $U \supseteq I^\star$ on B such that for every I^+-cover $C \subseteq A$ of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$.*

Proof: Let $\langle a_\delta |\ \delta < \kappa\rangle$ be an enumeration of A. For each $\delta < \kappa$ we define

$$A_\delta := \left\{\prod A' \;\middle|\; A' \subseteq \{a_\gamma |\ \gamma < \delta\} \cup \{1\}, A' \text{ is finite}\right\}.$$

Thus $\{a_\gamma |\ \gamma < \delta\} \subseteq A_\delta$ and $|A_\delta| < \kappa$.

CLAIM 1: *For each I^+-cover $C \subseteq A$ of B there exists a $\delta_C < \kappa$ such that $C \cap A_{\delta_C}$ is an I^+-cover of A_{δ_C}.*

Proof: Let $C \subseteq A$ be an I^+-cover of B. Let $\delta < \kappa$. For each $a \in A_\delta \cap I^+$ choose a $c_a \in C$ and a $\delta_a \geq \delta$ such that $a \cdot c_a \in I^+$ and $c_a \in A_{\delta_a}$. Then $\delta^\star := \sup\{\delta_a |\ a \in A_\delta \cap I^+\} < \kappa$. Let $\delta_0 := 0$, $\delta_{n+1} := \delta_n^\star$ and $\delta_C := \sup_{n\in\omega}\delta_n$.

Let $U_0 := \{\sum(C \cap A_{\delta_C}) |\ C \subseteq A \text{ is an } I^+\text{-cover of } B\}$.

CLAIM 2: *$U_0 \cup I^\star$ has the fip.*

Proof: Let $C_1, \ldots, C_n \subseteq A$ be I^+-covers of B. Let $\delta_i := \delta_{C_i}$. W.l.o.g. assume that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$. Since $1 \in A_{\delta_1} \cap I^+$, there is a $c_1 \in C_1 \cap A_{\delta_1}$ such that $c_1 \in I^+$. Since $c_1 \in A_{\delta_2} \cap I^+$ there exists a $c_2 \in C_2 \cap A_{\delta_2}$ such that $c_1 \cdot c_2 \in I^+$. At last we have choosen some $c_1 \in C_1 \cap A_{\delta_1}, \ldots, c_n \in C_n \cap A_{\delta_n}$ such that $c_1 \cdot c_2 \cdot \cdots \cdot c_n \in I^+$. Hence $\sum(C_1 \cap A_{\delta_1}) \cdot \cdots \cdot \sum(C_n \cap A_{\delta_n}) \in I^+$. Thus $U_0 \cup I^\star$ has the fip.

Now every ultrafilter $U \supseteq U_0 \cup I^\star$ has the required property. ∎

LEMMA 7: *Let $\kappa > \omega$ be regular, let I be a κ-complete, κ-dense ideal on some κ-complete Boolean algebra B. Then there is an ultrafilter $U \supseteq I^\star$ such that for each I^+-cover C of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$.*

Proof: Let $A \in [I^+]^{\leq\kappa}$ be dense in I^+. Lemma 6 gives us an uf $U \supseteq I^\star$ such that for each I^+-cover $C \subseteq A$ of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$.

U has the required property: Let $C \subseteq B$ be an I^+-cover of B. For each $a \in A$ choose $c_a \in C$, $d_a \in A$ and $y_a \in I$ such that $a \cdot c_a \in I^+$ and $d_a - y_a \leq a \cdot c_a$. $\{d_a |\ a \in A\}$ is dense in I^+, so it is a subset of A, which is an I^+-cover of B. Now we get an $A' \in [A]^{<\kappa}$ such that $\sum\{d_a |\ a \in A'\} \in U$. Then $\sum\{d_a - y_a |\ a \in A'\} \in U$ since I is κ-complete. Moreover $d_a - y_a \leq c_a$ implies $\sum\{c_a |\ a \in A'\} \in U$. This completes the proof. ∎

LEMMA 8: *Let $\kappa > \tau \geq \omega$, κ regular, let $I \subseteq \mathcal{P}(X)$ be a normal ideal on a nonempty set X such that $\{x \in X |\ \alpha \in x\} \in I^\star$ for all $\alpha < \kappa$. Suppose that $U \supseteq I^\star$ is an ultrafilter on X such that*

(1) *$\{x \in X \mid x \cap \kappa$ is $<\tau$closed$\} \in U$*
(2) *For each I^+-cover C of $\mathcal{P}(X)$ there is a $C' \in [C]^{<\kappa}$ with $\bigcup C' \in U$.*
Then U is non-(τ, κ)-regular.

Proof: Suppose that U is (τ, κ)-regular and that $\langle u_\alpha \mid \alpha < \kappa \rangle$ is a witness of the (τ, κ)-regularity. For each $x \in X$ let

$$b_x := \{\alpha < \kappa \mid x \in u_\alpha\}.$$

Then $|b_x| < \tau$. Thus the function $f\colon X \to \kappa$

$$f(x) := \bigcup (b_x \cap x)$$

is regressive on $c := \{x \in X \mid x \cap \kappa$ is $<\tau$-closed$\} \in U$. $\{f^{-1\prime\prime}\{\alpha\} \mid \alpha < \kappa\} \cup \{X \setminus c\}$ is an I^+-cover of $\mathcal{P}(X)$: If for some $b \in I^+$ $b \setminus c \in I$, then f is regressive on $b \cap c \in I^+$; since I is normal, there is an $\alpha \in rng\, f$ such that $b \cap c \cap f^{-1\prime\prime}\{\alpha\} \in I^+$. Now by (2) there is an $u \in U$ and an $\alpha < \kappa$ such that $f''u \subseteq \alpha$. Then $b_x \cap x \subseteq \alpha$ for all $x \in u$, so $\alpha \notin b_x$ for all $x \in u$ such that $\alpha \in x$, hence $x \notin u_\alpha$. Thus $u \cap u_\alpha \cap \{x \in X \mid \alpha \in x\} = \emptyset$, a contradiction. ∎

The proof of Lemma 9, the Boolean algebraic version of Lemma 8, is more technical.

LEMMA 9: *Let $\kappa > \tau \geq \omega$, κ regular, let B be a $(2^\kappa)^+$-complete, (κ^+, κ^+)-distributive Boolean algebra and let $I \subseteq B$ be an ideal, which is A-fine and A-normal for some $A = \langle a_\alpha \mid \alpha < \kappa \rangle \in {}^\kappa B$. Suppose that $U \supseteq I^*$ is an ultrafilter on B such that*
(1) $\prod_{\Gamma \in [\kappa]^{<\tau}} \left(\sum_{\gamma \in \Gamma} -a_\gamma + a_{\sup \Gamma} \right) \in U$
(2) *For each I^+-cover C of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$.*
Then U is non-(τ, κ)-regular.

Note that $\prod_{\Gamma \in [\kappa]^{<\tau}} \left(\sum_{\gamma \in \Gamma} -a_\gamma + a_{\sup \Gamma} \right) = \{x \in X \mid x \cap \kappa$ is $<\tau$-closed$\}$ if $B = \mathcal{P}(X)$ and $a_\gamma = \{x \in X \mid \gamma \in x\}$ for all $\gamma < \kappa$. Condition (1) is trivial if $\tau = \omega$.

Proof: Suppose that U is (τ, κ)-regular and that $\langle u_\alpha \mid \alpha < \kappa \rangle$ is a witness of the (τ, κ)-regularity. Let

$$c := \prod_{\Gamma \in [\kappa]^{<\tau}} \Big(\sum_{\gamma \in \Gamma} -a_\gamma + a_{\sup \Gamma} \Big).$$

W.l.o.g. for all $\alpha < \kappa$

$$u_\alpha \leq a_\alpha \cdot c.$$

For each $\alpha < \kappa$ let

$$b_\alpha := \prod_{\gamma<\alpha} \sum_{\beta>\gamma} u_\beta - \sum_{\beta>\alpha} u_\beta.$$

CLAIM 1: $b_\alpha \leq a_\alpha$ for all $\alpha < \kappa, \alpha \neq 0$.

Proof:

$$\begin{aligned}
b_\alpha &\leq \prod_{\gamma<\alpha} \sum_{\gamma<\beta\leq\alpha} u_\beta = \sum_{\substack{f:\, \alpha\to\alpha+1 \\ \forall\gamma f(\gamma)>\gamma}} \prod_{\gamma<\alpha} u_{f(\gamma)} \leq \sum_{\substack{\Gamma\in\mathcal{P}(\alpha+1) \\ \sup\Gamma=\alpha}} \prod_{\gamma\in\Gamma} u_\gamma \\
&= \sum_{\substack{\Gamma\in[\alpha+1]^{<\tau} \\ \sup\Gamma=\alpha}} \prod_{\gamma\in\Gamma} u_\gamma \text{ (regularity) } u_\alpha + \sum_{\substack{\Gamma\in[\alpha+1]^{<\tau} \\ \sup\Gamma=\alpha}} \prod_{\gamma\in\Gamma} u_\gamma \\
&\leq a_\alpha + \sum_{\substack{\Gamma\in[\alpha+1]^{<\tau} \\ \sup\Gamma=\alpha}} \prod_{\gamma\in\Gamma} (a_\gamma \cdot c) \\
&= a_\alpha + \Big(c \cdot \sum_{\substack{\Gamma\in[\alpha+1]^{<\tau} \\ \sup\Gamma=\alpha}} \prod_{\gamma\in\Gamma} a_\gamma\Big) \text{ } (\Gamma \neq \emptyset \text{ since } \alpha \neq 0) \\
&= a_\alpha + \Big(c \cdot \sum_{\substack{\Gamma\in[\alpha+1]^{<\tau} \\ \sup\Gamma=\alpha}} \big(\prod_{\gamma\in\Gamma} a_\gamma - a_\alpha\big)\Big) \\
&\leq a_\alpha + \Big(c \cdot \sum_{\Gamma\in[\alpha+1]^{<\tau}} \big(\prod_{\gamma\in\Gamma} a_\gamma - a_{\sup\Gamma}\big)\Big) \\
&= a_\alpha.
\end{aligned}$$

CLAIM 2: $\{b_\alpha |\, \alpha < \kappa\}$ is an I^+-cover of B.

Proof:

$$\begin{aligned}
\sum_{\alpha<\kappa} b_\alpha &= \sum_{\alpha<\kappa}\Big(\prod_{\gamma<\alpha}\sum_{\beta>\gamma} u_\beta - \sum_{\beta>\alpha} u_\beta\Big) = \sum_{\alpha<\kappa}\Big((-\sum_{\beta>\alpha} u_\beta) - \sum_{\gamma<\alpha}(-\sum_{\beta>\gamma} u_\beta)\Big) \\
&= \sum_{\alpha<\kappa}\Big(-\sum_{\beta>\alpha} u_\beta\Big) = -\prod_{\alpha<\kappa}\sum_{\beta>\alpha} u_\beta = -\sum_{\substack{f:\, \kappa\to\kappa \\ \forall\alpha f(\alpha)>\alpha}} \prod_{\alpha<\kappa} u_{f(\alpha)} = 1
\end{aligned}$$

since $\langle u_\alpha |\, \alpha < \kappa\rangle$ is a witness of the (τ, κ)-regularity. Using Claim 1 we get for every $d \in I^+$ $\sum_{\alpha<\kappa} a_\alpha b_\alpha d \geq a_0 d \sum_{\alpha<\kappa} b_\alpha = a_0 d \in I^+$. By A-normality there is some $\alpha < \kappa$ such that $b_\alpha d \in I^+$.

Now by (2) there exists a $\delta < \kappa$ such that $\sum_{\alpha<\delta} b_\alpha \in U$. Thus $u_\delta \cdot \sum_{\alpha<\delta} b_\alpha \in U$, but

$$u_\delta \cdot \sum_{\alpha<\delta} b_\alpha = u_\delta \cdot \sum_{\alpha<\delta} \Big(\prod_{\gamma<\alpha} \sum_{\beta>\gamma} u_\beta - \sum_{\beta>\alpha} u_\beta \Big) \leq u_\delta \cdot \sum_{\alpha<\delta} \Big(- \sum_{\beta>\alpha} u_\beta \Big) = 0.$$

Contradiction. ∎

Now part (1) of the main theorem follows from Lemma 7 and 8. Lemma 7 and 9 imply Theorem 10, which is a generalisation of part (1) of the main theorem.

THEOREM 10: *Let $\kappa > \omega$ be regular, let B be a $(2^\kappa)^+$-complete, (κ^+, κ^+)-distributive Boolean algebra. Suppose that $I \subseteq B$ is a κ-complete, κ-dense ideal, which is A-fine and A-normal for some $A = \langle a_\alpha \mid \alpha < \kappa \rangle \in {}^\kappa B$. Then there is an ultrafilter $U \supseteq I^\star$ on B, which is non-(τ, κ)-regular for all $\tau < \kappa$ such that $\prod_{\Gamma \in [\kappa]^{<\tau}} \left(\sum_{\gamma\in\Gamma} -a_\gamma + a_{\sup\Gamma} \right) \in I^\star$.*

We can use the previous lemmas to estimate the cardinality of some ultrapowers. Laver [L] proved, that every ultrafilter U on ω_1, which is generated by a normal filter and some set of size ω_1, is non-(ω, ω_1)-regular. Moreover $|\omega^{\omega_1}/U| = \aleph_1$ if CH holds. Actually Laver's argument shows: Let U be a non-(ν, ν^+)-regular ultrafilter on ν^+, which is generated by a ν^+-complete filter and some set of size 2^ν. Then $|\nu^{\nu^+}/U| \leq 2^\nu$.

COROLLARY 11: *Let $I \subseteq \mathcal{P}(\nu^+)$ be a normal, ν^+-dense ideal on ν^+, ν regular, such that $\nu^+ \subseteq I$. Then there is an ultrafilter $U \supseteq I^\star$ on ν^+ such that $|\nu^{\nu^+}/U| \leq 2^\nu$.*

Proof: I is ν^+-complete. Lemma 7 gives us an ultrafilter $U \supseteq I^\star$ on ν^+ such that for each I^+-cover C of $\mathcal{P}(\nu^+)$ there is a $C' \in [C]^{\leq\nu}$ with $\bigcup C' \in U$. By Lemma 8 U is non-(ν, ν^+)-regular, since $\{\alpha < \nu^+ \mid \mathrm{cf}(\alpha) = \nu\} \in I^\star$ (Shelah [Sh]). Let $A \in [I^+]^{\leq\nu^+}$ be dense in $\mathcal{P}(\nu^+)/I$. Then $I^\star \cup \{\bigcup A' \mid A' \in [A]^{\leq\nu}, \bigcup A' \in U\}$ generates U, since for any $b \subseteq \nu^+$ the set $C := \{a \in A \mid a \cap b \in I \text{ or } a \setminus b \in I\}$ is an I^+-cover of $\mathcal{P}(\nu^+)$. Thus U is generated by $I^\star$ and some set of size 2^ν. Then by Laver's argument $|\nu^{\nu^+}/U| \leq 2^\nu$. ∎

If $\nu = \omega$, then $|\omega^{\omega_1}/U| = 2^{\aleph_0}$ for the following reason: U is not ω_1-complete. Let $\langle u_n \mid n < \omega \rangle \in {}^\omega U$ be decreasing such that $\bigcap_{n<\omega} u_n = \emptyset$. For every $A \subseteq \omega$ let $f_A \colon \omega_1 \to [\omega]^{<\omega}$, $f_A(x) := \{n \in A \mid x \in u_n\}$. Then $A \mapsto [f_A]_U$ is an injection from $\mathcal{P}(\omega)$ into $([\omega]^{<\omega})^{\omega_1}/U$ because $f_A(x) \neq f_B(x)$ for all $x \in u_n$ if $n \in A \setminus B$.

Layered ideals and non-regular ultrafilters

Before we prove part (2) of the main theorem we need two more lemmas.

LEMMA 12: *Suppose that B_0 and B are κ-complete Boolean algebras, B_0 is a regular subalgebra of B and $U \subseteq B_0$ is an ultrafilter in B_0 such that for every B_0^+-cover $C \subseteq B_0$ of B_0 there exists a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$. If $A \subseteq B$ is a B^+-cover of B, then $\{\sum A' | \; A' \in [A]^{<\kappa}\}$ is an U^+-cover of B.*

Proof: Let $A \subseteq B$ be a B^+-cover of B, let $d \in U^+$. Consider $A_0 := \{b \in B_0 | \; b \cdot d = 0$ or for some $a \in A$ b is a projection of $d \cdot a$ in $B_0\}$.

CLAIM 1: A_0 is a B_0^+-cover of B_0.

Proof: Let $b_0 \in B_0^+$. If $b_0 \cdot d = 0$, then $b_0 \in A_0$. If $b_0 \cdot d \neq 0$, then there is an $a \in A$ such that $a \cdot b_0 \cdot d \neq 0$. Let $b \in B_0$ be a projection of $a \cdot b_0 \cdot d$. Then $b \cdot a \cdot b_0 \cdot d \neq 0$, particularly $b \cdot b_0 \neq 0$. Moreover $b \in A_0$ since b is also a projection of $a \cdot d$. So in both cases there is a $b \in A_0$ such that $b \cdot b_0 \neq 0$.

By the assumption there is an $A_0' \in [A_0]^{<\kappa}$ with $\sum A_0' \in U$. Either $\sum\{b \in A_0' | \; b \cdot d = 0\} \in U$ or $\sum\{b \in A_0' |$ for some $a \in A$ b is a projection of $d \cdot a\} \in U$. Since $d \cdot \sum\{b \in A_0' | \; b \cdot d = 0\} = 0$ the first case is impossible ($d \in U^+$). So w.l.o.g. for every $b \in A_0'$ there is an $a \in A$ such that b is a projection of $d \cdot a$.

Choose $A' \in [A]^{<\kappa}$ such that for each $b \in A_0'$ there exists such an $a \in A'$. Let $a^\star := \sum A'$.

CLAIM 2: $a^\star \cdot d \in U^+$.

Proof: Let $u \in U$. $u \cdot \sum A_0' \in U$ since $\sum A_0' \in U$. So there is an $a_0 \in A_0'$ such that $u \cdot a_0 \neq 0$. Choose an $a \in A'$ such that a_0 is a projection of $a \cdot d$. Then $u \cdot a_0 \cdot a \cdot d \neq 0$ (since $u \cdot a_0 \in B_0^+$) and therefore $u \cdot \sum A' \cdot d \neq 0$, i.e. $u \cdot a^\star \cdot d \neq 0$.

Claim 2 completes the proof of Lemma 12. ∎

LEMMA 13: *Let $\kappa > \omega$ be regular, let B_0, B and U be as in Lemma 12 and assume that $|B| \leq \kappa$. Then there exists an ultrafilter $V \supseteq U$ on B such that for each B^+-cover C of B there is a $C' \in [C]^{<\kappa}$ such that $\sum C' \in V$.*

Proof: Using Lemma 6 (with $A := B$, $I :=$ the ideal generated by $U^\star$ in B) we get an ultrafilter $V \supseteq U$ on B such that for each U^+-cover C of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in V$. If A is a B^+-cover of B, then by Lemma 12 $C := \{\sum A' | \; A' \in [A]^{<\kappa}\}$ is a U^+-cover of B. Hence there exists a $C' \in [C]^{<\kappa}$ with $\sum C' \in V$ and therefore an $A'' \in [A]^{<\kappa}$ with $\sum A'' \in V$ (κ is regular). ∎

LEMMA 14: *Let $\kappa > \omega$ be regular. Suppose $\square_\kappa$ and I is a κ-complete, strongly κ-layered ideal on a κ-complete Boolean algebra B. Then there is an ultrafilter $U \supseteq I^\star$ such that for each I^+-cover C of B there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U$.*

Proof: Let $\langle B_\alpha | \ \alpha < \kappa^+\rangle$ be a continuous increasing chain of Bas such that $B/I = \bigcup_{\alpha<\kappa^+} B_\alpha$ and for all $\alpha < \kappa^+$ such that $\mathrm{cf}(\alpha) = \kappa$ B_α is a κ-complete regular subalgebra of B/I of cardinality κ. Let $\langle \mathcal{C}_\alpha | \ \alpha \in \kappa^+ \cap \mathrm{Lim}\rangle$ be a $\square_\kappa$-sequence, i.e.

(i) $\mathcal{C}_\alpha$ is club in α

(ii) $\mathcal{C}_\beta = \mathcal{C}_\alpha \cap \beta$ if β is a limit point of $\mathcal{C}_\alpha$

(iii) $ot(\mathcal{C}_\alpha) < \kappa$ if $\mathrm{cf}(\alpha) < \kappa$.

Thus $ot(\mathcal{C}_\alpha) = \kappa$ if $\mathrm{cf}(\alpha) = \kappa$. Let $\langle \alpha_i | \ i < \kappa^+\rangle$ be the strictly increasing enumeration of $\{\delta < \kappa^+ | \ \mathrm{cf}(\delta) = \kappa\}$. Choose for each $i < \kappa^+$ an enumeration $\langle b_i^\delta | \ \delta < \kappa\rangle$ of B_i. For each $j \in \kappa^+ \cap \mathrm{Lim}$ and each $i \in \mathcal{C}_j \cup \{j\}$ let

$$B_j^i := \left\{ \sum A \;\middle|\; A \subseteq \{b_{\alpha_k}^\delta \,|\, \delta < ot(i \cap \mathcal{C}_j),\ k \in i \cap \mathcal{C}_j\},\quad A \text{ is finite} \right\}.$$

We notice the following facts for all $j, j' \in \kappa^+ \cap \mathrm{Lim}$ and all $i, i' \in \mathcal{C}_j \cup \{j\}$:

(1) $|B_j^i| < \kappa$ if $\mathrm{cf}(j) < \kappa$ or $i < j$

(2) $B_j^i \subseteq B_j^{i'}$ if $i \le i'$

(3) $\bigcup_{i \in \mathcal{C}_j} B_j^i = B_{\alpha_j}$ if $\mathrm{cf}(j) = \kappa$

(4) $B_j^i \subseteq B_{\alpha_i} \subseteq B_{\alpha_j}$

(5) $B_j^i = B_{j'}^i$ if j is a limit point of $\mathcal{C}_{j'}$ (since $i \cap \mathcal{C}_j = i \cap \mathcal{C}_{j'}$)

(6) $\bigcup_{k \in i \cap \mathcal{C}_j} B_j^k = B_j^i$ if i is a limit point of $\mathcal{C}_j$

(7) $\{\sum A | \ A \in [B_j^i]^{<\omega}\} \subseteq B_j^i$.

Now we define recursively a sequence $\langle U_i | \ i < \kappa^+\rangle$ which satisfies the following conditions:

(a) U_i is an uf on B_{α_i} such that for each $B_{\alpha_i}^+$-cover $C \subseteq B_{\alpha_i}$ of B_{α_i} there is a $C' \in [C]^{<\kappa}$ with $\sum C' \in U_i$

(b) $U_i \subseteq U_j$ for all $i < j < \kappa^+$

(c) $V_j \subseteq U_j$ for all $j \in \kappa^+ \cap \mathrm{Lim}$ such that $\mathrm{cf}(j) < \kappa$

where

$$V_j := \left\{ \sum D \;\middle|\; D \subseteq B_j^j \text{ is a } U_k^+\text{-cover of } B_j^j \text{ for each } k \in \mathcal{C}_j \right\}.$$

Let $j < \kappa^+$ and suppose that $\langle U_i | \ i < j\rangle$ is already defined as required.

CASE 1: Let $j = 0$. Using Lemma 6 we get an ultrafilter U_0 on B_{α_0} satisfying (a).

CASE 2: Let $j \in \mathrm{Succ}$, $j = k+1$. Lemma 13 gives us an ultrafilter $U_j \supseteq U_k$ on B_{α_j}, which satisfies (a).

CASE 3: Let $j \in \mathrm{Lim}$, $\mathrm{cf}(j) < \kappa$. For each $\delta < \kappa$ let

$$A^{\delta}_{\alpha_j} := \Big\{ \sum A \;\Big|\; A \subseteq \{b^{\beta}_{\alpha_j} \mid \beta < \delta\} \cup B^j_j, A \text{ is finite} \Big\}.$$

Then $|A^{\delta}_{\alpha_j}| < \kappa$ (since $|B^j_j| < \kappa$ if $\mathrm{cf}(j) < \kappa$) and $\bigcup_{\delta<\kappa} A^{\delta}_{\alpha_j} = B_{\alpha_j}$. For each $B^+_{\alpha_j}$-cover $C \subseteq B_{\alpha_j}$ of B_{α_j} such that $\{\sum C' \mid C' \in [C]^{<\kappa}\} \subseteq C$ there exists a $\delta_C < \kappa$ such that $C \cap A^{\delta_C}_{\alpha_j}$ is a U^+_k-cover of $A^{\delta_C}_{\alpha_j}$ for each $k \in \mathcal{C}_j$: Let $\delta_0 := 0$. If $\delta_n < \kappa$ is already defined, choose for every $k \in \mathcal{C}_j$ and every $b \in A^{\delta_n}_{\alpha_\gamma} \cap U^+_k$ some $c^k_b \in C$ and some $\delta^k_b > \delta_n$ such that $b \cdot c^k_b \in U^+_k$ and $c^k_b \in A^{\delta^k_b}_{\alpha_j}$ (this is possible by Lemma 12). Let $\delta_{n+1} := \sup\{\delta^k_b \mid k \in \mathcal{C}_j, b \in A^{\delta_n}_{\alpha_j} \cap U^+_k\}$ and $\delta_C := \sup_{n\in\omega}\delta_n$. Then $C \cap A^{\delta_C}_{\alpha_j}$ is a U^+_k-cover of $A^{\delta_C}_{\alpha_j}$ for each $k \in \mathcal{C}_j$. Particularly $C \cap A^{\delta_C}_{\alpha_j}$ is a U^+_k-cover of B^j_j for each $k \in \mathcal{C}_j$ since $B^j_j \subseteq A^{\delta_C}_{\alpha_j}$. Let

$$W_j := \Big\{ \sum (C \cap A^{\delta_C}_{\alpha_j}) \;\Big|\; C \subseteq B_{\alpha_j} \text{ is a } B^+_{\alpha_j}\text{-cover of } B_{\alpha_j} \text{ such that } \{\sum C' \mid C' \in [C]^{<\kappa}\} \subseteq C \Big\}.$$

CLAIM A: $\bigcup_{i<j} U_i \cup V_j \cup W_j$ has the fip.

Proof: Suppose that $D_1, ..., D_m \subseteq B^j_j$ are U^+_k-covers of B^j_j for each $k \in \mathcal{C}_j$, $C_1, ..., C_n \subseteq B_{\alpha_j}$ are $B^+_{\alpha_j}$-covers of B_{α_j} such that $\{\sum C' \mid C' \in [C_p]^{<\kappa}\} \subseteq C_p$, $1 \le p \le n$, and let $u \in \bigcup_{i<j} U_i$. There is an $k \in \mathcal{C}_j$ such that $u \in U_k$. Let $\delta_p := \delta_{C_p} (1 \le p \le n)$. It suffices to show that

$$\sum D_1 \cdot \dots \cdot \sum D_m \cdot \sum (C_1 \cap A^{\delta_1}_{\alpha_j}) \cdot \dots \cdot \sum (C_n \cap A^{\delta_n}_{\alpha_j}) \in U^+_k.$$

W.l.o.g. $\delta_1 \le \delta_2 \le \dots \le \delta_n$. Since $1 \in B^j_j \cap U^+_k$ there is a $d_1 \in D_1$ such that $d_1 \in U^+_k$. Since $d_1 \in B^j_j \cap U^+_k$ there is a $d_2 \in D_2$ such that $d_1 \cdot d_2 \in U^+_k$. Since $d_1 \cdot d_2 \in B^j_j \cap U^+_k$ there is a $d_3 \in D_3$ such that $d_1 \cdot d_2 \cdot d_3 \in U^+_k$. At last we have choosen $d_1 \in D_1, \dots, d_m \in D_m$ such that $d_1 \cdots d_m \in B^j_j \cap U^+_k$. Since $C_1 \cap A^{\delta_1}_{\alpha_j}$ is a U^+_k-cover of B^j_j there is a $c_1 \in C_1 \cap A^{\delta_1}_{\alpha_j}$ such that $d_1 \cdots d_m \cdot c_1 \in U^+_k$. Since $d_1 \cdots d_m \cdot c_1 \in A^{\delta_2}_{\alpha_j} \cap U^+_k$ there is a $c_2 \in C_2 \cap A^{\delta_2}_{\alpha_j}$ such that $d_1 \cdots d_m \cdot c_1 \cdot c_2 \in U^+_k$.

At last we have choosen $c_1 \in C_1 \cap A^{\delta_1}_{\alpha_j}, \ldots, c_n \in C_n \cap A^{\delta_n}_{\alpha_j}$ such that $d_1 \cdots d_m \cdot c_1 \cdots c_n \in U^+_k$. Hence $\sum D_1 \cdots \sum D_m \cdot \sum(C_1 \cap A^{\delta_1}_{\alpha_j}) \cdots \sum(C_n \cap A^{\delta_n}_{\alpha_j}) \in U^+_k$.

Let $U_j \supseteq \bigcup_{i<j} U_i \cup V_j \cup W_j$ be an arbitrary uf on B_{α_j}. Then U_j satisfies the covering property (a) since $W_j \subseteq U_j$.

CASE 4: Let $j \in \mathrm{Lim}$, $\mathrm{cf}(j) = \kappa$. Then $\alpha_j = \sup_{i<j}\alpha_i$ and $B_{\alpha_j} = \bigcup_{i<j} B_{\alpha_i}$. Let $U_j := \bigcup_{i<j} U_i$. U_j is an uf on B_{α_j}.

CLAIM B: U_j satisfies (a).

Proof: Let $C \subseteq B_{\alpha_j}$ be a $B^+_{\alpha_j}$-cover of B_{α_j}. W.l.o.g. $\{\sum C' \mid C' \in [C]^{<\kappa}\} \subseteq C$. Let $i_0 := \min \mathcal{C}_j$. If $i_n \in \mathcal{C}_j$ is already defined, then choose for each $k \in i_n \cap \mathcal{C}_j$ and each $b \in B^{i_n}_j \cap U^+_k$ a $c^k_b \in C$ and an $i^k_b > i_n$ such that $c^k_b \cdot b \in U^+_k, i^k_b \in \mathcal{C}_j$ and $c^k_b \in B^{i^k_b}_j$ (this is possible by Lemma 12). Choose $i_{n+1} \in \mathcal{C}_j$, $i_{n+1} \geq \sup\{i^k_b \mid k \in i_n \cap \mathcal{C}_j,\ b \in B^{i_n}_j \cap U^+_k\}$. Let $i := \sup_{n\in\omega} i_n$. Then $i \in j \cap \mathrm{Lim}$ and $\mathrm{cf}(i) < \kappa$ since $i_n < i_{n+1} < j$. i is a limit point of $\mathcal{C}_j$, so $\mathcal{C}_i = i \cap \mathcal{C}_j$ and $B^i_i = B^i_j$. Let $D := C \cap B^i_i$. D is a U^+_k-cover of B^i_i for each $k \in \mathcal{C}_i$: For each $k \in \mathcal{C}_i$ and each $b \in B^i_i \cap U^+_k$ $(= B^i_j \cap U^+_k)$ there is a $n < \omega$ such that $k \in i_n \cap \mathcal{C}_j$ and $b \in B^{i_n}_j \cap U^+_k$. Hence there is a $c^k_b \in C \cap B^{i_{n+1}}_j$ such that $b \cdot c^k_b \in U^+_k$. Thus $c^k_b \in C \cap B^i_i = D$ since $B^{i_{n+1}}_j \subseteq B^i_j = B^i_i$. So $\sum D \in V_i$ and therefore $\sum D \in U_j$.

This completes the definition of $\langle U_i \mid i < \kappa^+\rangle$.

Now let $V := \bigcup_{i<\kappa^+} U_i$. V is an uf on B. For each B^+-cover $C \subseteq B$ of B there exists an $i < \kappa^+$ such that $C \cap B_{\alpha_i}$ is a $B^+_{\alpha_i}$-cover of B_{α_i}. Thus there is a $C' \in [C \cap B_{\alpha_i}]^{<\kappa}$ with $\sum C' \in U_i$. Therefore $\sum C' \in V$. Let $U := \bigcup V$. Then $U \supseteq I^\star$ and for each I^+-cover C of B there is a $C' \in [C]^{<\kappa}$ such that $\sum C' \in U$. This completes the proof of Lemma 14. ∎

Part (2) of the main theorem follows from Lemma 8 and 14. Lemma 9 and 14 imply Theorem 15, which is a generalization of part (2) of the main theorem.

THEOREM 15: *Let $\kappa > \omega$ be regular, let B be a $(2^\kappa)^+$-complete, (κ^+, κ^+)-distributive Boolean algebra. Suppose $\Box_\kappa$ and $I \subseteq B$ is a κ-complete, strongly κ-layered ideal, which is A-fine and A-normal for some $A = \langle a_\alpha \mid \alpha < \kappa\rangle \in {}^\kappa B$. Then there is an ultrafilter $U \supseteq I^\star$ on B, which is non-(τ,κ)-regular for all $\tau < \kappa$ such that $\prod_{\Gamma\in[\kappa]^{<\tau}} \left(\sum_{\gamma\in\Gamma} -a_\gamma + a_{\sup\Gamma}\right) \in I^\star$.*

We give an application of the main theorem to limit cardinals.

COROLLARY 16: *It is consistent relative to the existence of a measurable cardinal, that there is an uniform ultrafilter on a weakly compact, non-measurable cardinal κ, which is non-(τ, κ)-regular for all $\tau < \kappa$.*

Proof: Starting with a measurable cardinal, Kunen and Paris ([KP], Theorem 4.4) constructed a generic extension with a κ-complete, κ^+-saturated, normal, uniform ideal I on a weakly compact, non-measurable cardinal κ. Looking at the construction in Lemma 4.9 of [KP], one can see, that this ideal is actually κ-dense and $\{\alpha < \kappa | \operatorname{cf}(\alpha) \geq \tau\} \in I^\star$ for all $\tau < \kappa$. Using our main theorem, there is an uniform ultrafilter on κ, which is non-(τ, κ)-regular for all $\tau < \kappa$. ∎

The main theorem may also become a tool to get non-(τ, κ)-regular ultrafilters even on sets X such that $|X| > \kappa$, if it becomes possible to construct suitable ideals.

Remark 17: The main theorem gives a new proof, that under $\mathrm{MA}_{\aleph_1}$ there is no ω_1-dense, ω_1-complete ideal on ω_1 (see [FMSh1] or [T]), since Laver [L] showed, that under $\mathrm{MA}_{\aleph_1}$ all uniform ultrafilters on ω_1 are regular. ∎

References

[BSV] B. Balcar, P. Simon and P. Vojtáš, *Refinement properties and extensions of filters in Boolean algebras,* Trans. Amer. Math. Soc. **267** (1981), 265–283.

[D] H.-D. Donder, *Regularity of ultrafilters and the core model,* Israel J. Math. **63** (1988), 289–322.

[DJK] H.-D. Donder, R. B. Jensen and B. J. Koppelberg, *Some applications of the core model,* in *Set Theory and Model Theory, Proc. Bonn* 1979, Lecture Notes in Mathematics **872**, Springer-Verlag, Berlin, 1981, pp. 55–97.

[FMSh1] M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals and non-regular ultrafilters. Part I,* Ann. of Math. **127** (1988), 1–47.

[FMSh2] M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals and non-regular ultrafilters. Part II,* Ann. of Math. **127** (1988), 521–545.

[H] M. Huberich, *Large ideals on small cardinals,* Ann. Pure Appl. Logic. **63** (1993).

[Ka] A. Kanamori, *Weakly normal filters and irregular ultrafilters,* Trans. Amer. Math. Soc. **220** (1976), 393–399.

[Ke] J. Ketonen, *Nonregular ultrafilters and large cardinals,* Trans. Amer. Math. Soc. **224** (1976), 61–73.

[KP] K. Kunen and J. B. Paris, *Boolean extensions and measurable cardinals,* Ann. Math. Logic **2** (1971), 359–377.

[L] R. Laver, *Saturated ideals and nonregular ultrafilters,* in *Patras Logic Symposium* (G. Metakides, ed.), North-Holland, Amsterdam, 1982, pp. 297–305.

[P] K. Prikry, *On a problem of Gillman and Keisler*, Ann. Math. Logic **2** (1970), 179–187.

[Sh] S. Shelah, *Around Classification Theory of Models,* Lecture Notes in Math. **1182**, Springer-Verlag, Berlin, 1986.

[T] A. Taylor, *On saturated sets of ideals and Ulam's problem,* Fund. Math. **109** (1980), 37–53.